
REPRODUCING THE LPF FULL DYNAMICS
CALIBRATION VIA THE IRLS METHOD

COLLEGE OF ENGINEERING
BRAD NICHOLAS RATTO
RATTOB@MY.ERAU.EDU

FACULTY SPONSOR
DR. DANIELE VETRUGNO
DANIELE.VETRUGNO@UNITN.IT

PROJECT PERIOD
SUMMER 2021

*University of Florida
Gainesville, Florida*



Acknowledgments

I would like to thank the University of Florida Gainesville for hosting the International Research Experiences for Undergraduates (IREU), along with the IREU's coordinators Dr. Paul Fulda and Dr. Peter Wass. A special thanks to research mentor Dr. Daniele Vetrugno for guiding me and providing with the resources necessary. I would like to thank graduate students Nathaniel Strauss, Jonathon Baird and Quang Nam.

I would like to thank the National Science Foundation (NSF) for funding programs such as the IREU at University of Florida, NSF grant numbers include NSF PHY-1950830 and NSF PHY-1460803



Contents

1	Introduction	1
2	Project Background	4
2.1	LISA Pathfinder Dynamics	4
3	Project Approach	8
3.1	Iterative Reweighed Least Squares	9
4	Conclusion	11



1 Introduction

The detection of Gravitational Waves (GWs) has revealed yet another method in which information is transmitted across the cosmos, bringing with it further insight into the inner workings of our universe. In order to detect such phenomena, laser interferometers are used to measure microscopic deformations in space-time caused by transient GWs; ultimately granting us the ability to traverse ever further into the new frontier of gravitational wave based astronomy. Although serving as a valuable apparatus for making detections, it is important to realize that these ground-based laser interferometers are affected by seismic noise and shifts in the local gravity gradient which limits the ability to detect events in the mHz range. As eluded to, the development of space-based laser interferometers opens the possibility to detect events which produce GW emissions in the mHz regime, such events include the formation and coalescence of super-massive black holes.

One such example of a future space-based interferometer is the Laser Interferometer Space Antenna (LISA) project, consisting of a constellation of three separate spacecraft trailing Earth's orbit, LISA aims to obtain both information regarding the polarization of emitted GWs while simultaneously measuring source parameters of astrophysical significance. As a result, the LISA Pathfinder (LPF) mission was launched as a method to validate performance requirements and prove the efficacy of the technology to be later used in the LISA mission. The technology package on board the LPF satellite was comprised of two main components, the Gravitational Reference Sensor (GRS), which consist of two free falling masses contained within an electrode housing maintained under vacuum, and the Optical Metrology System (OMS), which is comprised of a set of heterodyne laser interferometers.

A schematic of the LPF satellite is shown in Figure 1 and will be used to summarize the objectives and highlight key principles of the mission. As such, the main objective for the LPF mission was to measure the relative displacement, $\Delta\mathbf{x}(t)$, of the two test masses, TM_1 and TM_2 , in order to quantify the rate of fluctuation of the differential stray force per unit mass $\Delta\mathbf{g}(t)$, or as commonly refer to as the differential acceleration noise. Requirements for the fluctuations in the acceleration noise where not to exceed a level of $30 \text{ fm s}^{-2}/\sqrt{\text{Hz}}$; not only was the requirement satisfied but it was shown that the acceleration noise reached levels as low as $S_{\Delta\mathbf{g}}^{1/2} \cong 1.74 \pm 0.05 \text{ fm s}^{-2}/\sqrt{\text{Hz}}$ above 2 mHz and $S_{\Delta\mathbf{g}}^{1/2} \cong (6 \pm 1) \times 10 \text{ fm s}^{-2}/\sqrt{\text{Hz}}$ at 20 μHz .

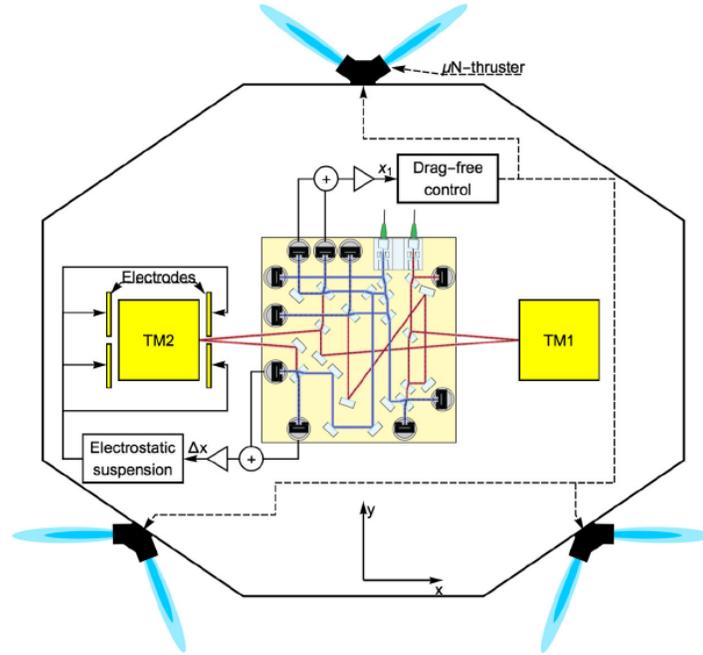


Figure 1: A schematic of the LPF satellite and the scientific payload is shown above [1]. The scientific payload is comprised of two test-masses within their vacuum enclosures and surrounded by force sensing actuators or commonly known as the Gravitational Reference Sensor (GRS). An optical table with a set of heterodyne laser interferometers form the Optical Metrology System (OMS), it's purpose is two-fold. The first of which is to measure the differential position along the x-axis of the two test masses along with their differential angular positions with respect to each other. The second is to measure the position and angular orientation of TM_1 with respect to the spacecraft body frame. These interferometric measurements are used as inputs to the Drag-Free and Suspension control loops; the former of which is used to keep the spacecraft's position fixed with TM_1 via the use of the μN thrusters and the latter control loop keeps the position of TM_2 fixed to TM_1 via the force sensing capacitors of the GRS.

By the very nature of the experiment, the LPF satellite, along with it's scientific payload, behaves as a three-body system; in order to ensure the stability of this system, the LPF satellite was equipped with force sensing actuators and μN thrusters controlled via Suspension and Drag-Free control loops. The purpose of these mechanisms are to keep both the spacecraft body and TM_2 locked in position with respect



to TM_1 . As previously mentioned, the OMS is comprised of two heterodyne laser interferometers; this optical configuration serves two purposes. The first was to sense the differential length, along the x-axis, and differential angles of TM_1 and TM_2 with respect to each other, see Figure 2 for the axes convention used. The second purpose was to measure the position and angle of TM_1 with respect to the spacecraft body frame. These interferometric measurements are used as inputs to control loops to keep the spacecraft body and TM_2 locked to TM_1 .

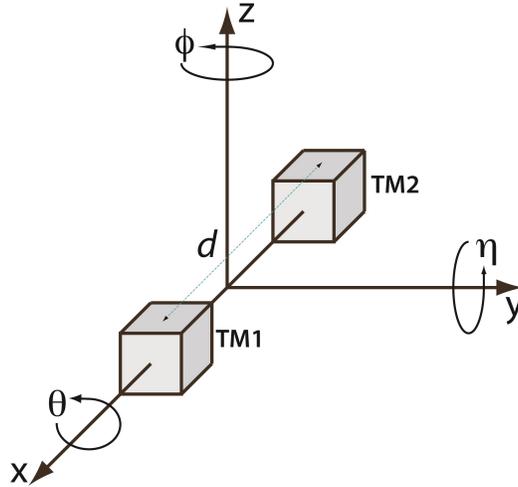


Figure 2: The axis used for both test masses and the spacecraft follows the right hand rule, the x-axis is defined as the long sensitive axis and is set to be orthogonal to both TM_1 and TM_2 and passes through the OMS [4]. As such, the differential displacement between the two masses is defined as $d \equiv \Delta x(t) = x_2(t) - x_1(t)$. The differential angles between the two test masses are then defined as $\Delta \eta(t) = \eta_2(t) - \eta_1(t)$ and $\Delta \phi(t) = \phi_2(t) - \phi_1(t)$. The final measurement is with respect to TM_1 and the spacecraft frame, or simply defined as the position, x_1 , of TM_1 and the corresponding angular position, η_1 and ϕ_1 .



2 Project Background

The project had two objectives, the first (1) was to understand and recreate the Iterative Reweighted Least Squares (IRLS) technique developed by the LISA collaboration to identify the parameters as specified by the LPF dynamics model, this is later used to assess the fidelity and accuracy of the model. The second objective (2) is a case study to investigate the impact of modifying the set of parameters specified to be solved for by the IRLS algorithm; the results of which could be used to better understand and account for cross-talk. It is for these reasons that an understanding of the dynamics of LPF is necessary.

2.1 LISA Pathfinder Dynamics

As previously mentioned, the nature of the LPF satellite, along with its scientific payload, form a three-body system, this includes the spacecraft body, TM_1 , and TM_2 , whose dynamics can be described by the following equation of motion (EOM):

$$\Delta g_x(t) = \Delta \ddot{x}(t) - g_c(t) + \omega_2^2 \Delta x(t) + \Delta \omega_{12}^2 x_1(t) \quad (1)$$

The EOM in equation 1 has four main components, the $\Delta g_x(t)$ term which is the sought after differential acceleration noise prior to correcting for inertial forces, $\Delta \ddot{x}(t)$ and $\Delta x(t)$ which is the measured differential acceleration and differential position between the two test-masses respectively, the $g_c(t)$ term which represents the command forces induced onto the test masses and finally the ω_2^2 and $\Delta \omega_{12}^2$ terms are used to couple the relative motion of the test masses and spacecraft. The command force $\Delta g_c(t)$ is defined as follows:

$$g_c(t) = \lambda_2 \frac{F_{x_2}}{m_{TM_2}}(t - \tau_2) - \lambda_1 \frac{F_{x_1}}{m_{TM_1}}(t - \tau_1) = \lambda_2 f_{x_2}(t - \tau_2) - \lambda_1 f_{x_1}(t - \tau_1) \quad (2)$$

Where f_{x_1} and f_{x_2} are the applied forces on each test mass, with masses m_{TM_1} and m_{TM_2} , while F_{x_1} and F_{x_2} are the same applied forces per unit mass. The time delay coefficients τ_1 and τ_2 are used to account for the delay between the OMS and



the force-sensing actuators of the GRS. Finally, gain coefficients, λ_1 and λ_2 are the parameters to be solved for that account for discrepancies in the response of the force-sensing actuators. By substituting equation (2) into equation (1) we obtain:

$$\Delta g_x(t) = \Delta \ddot{x}(t) + \lambda_2 f_{x_2}(t - \tau_2) - \lambda_1 f_{x_1}(t - \tau_1) + \omega_2^2 \Delta x(t) + \Delta \omega_{12}^2 x_1(t) \quad (3)$$

It is important to note that the time delay coefficients, τ_1 and τ_2 , introduce non-linear effects, however we can linearize the dynamics with a set of linearized delay coefficients.

$$\lambda_j f_{x_j}(t - \tau_j) = \lambda_j f_{x_j}(t) - \lambda_j \tau_j \dot{f}_{x_j}(t) = \lambda_j f_{x_j} - C_j \dot{f}_{x_j} \quad (4)$$

It is now possible to rewrite equation (3) in the following form:

$$\begin{aligned} \Delta g_x(t) = \Delta \ddot{x}(t) + \lambda_1 f_{x_1}(t) - \lambda_2 f_{x_2}(t) - C_1 \dot{f}_{x_1}(t) \\ + C_2 \dot{f}_{x_2}(t) + \omega_2^2 \Delta x(t) + \Delta \omega_{12}^2 x_1(t) \end{aligned} \quad (5)$$

The environment of spacecraft is such that there exists non-zero forces which act upon TM_1 and TM_2 ; these forces are a result of gravitational effects and those induced by magnetic and/or electrostatic fields. Much like the spring constant for a spring-mass system, these forces act on the test masses as a result of their relative motion; this effect is captured by the stiffness and differential stiffness terms seen in equation (1) as ω_j^2 and $\Delta \omega_{12}^2$ respectively. The individual stiffness terms are defined as follows:

$$\omega_j^2 = \omega_{j,0}^2 + \alpha_{x_j} F_{\max,j} + \alpha_{\phi_j} N_{\max,j} \quad (6)$$

Where the $\omega_{j,0}^2$ term refers to the background stiffness which is used to capture the gravitational effects and capacitive sensing basis which remains constant through the duration of the experiment. The force-sensing actuators have maximum Torque and Force limits applied, hence $F_{\max,j}$ and $N_{\max,j}$. It is important to note that, regardless of the values specified for maximum force and torque, the force-sensing actuators are set to maintain LPF at a constant stiffness. Parameters α_{x_j} and α_{ϕ_j} couple with the



torque authorities to give the stiffness components along the x-axis relative to their axis of influence; where α_{x_j} is written as:

$$\alpha_{x_j} = -\lambda_V^2 \frac{1}{m_{\text{TM}_j}} \frac{\frac{\partial^2 C_X^*}{\partial x^2}}{\frac{\partial C_X^*}{\partial x}} \quad (7)$$

Where $C_X^* \equiv C_X + C_{X,h}$ is the total X electrode capacitance, where the first term is the capacitance of the electrode relative to the test mass, and the second term refers to the capacitance of the electrode with respect to grounding plates. The λ_V^2 term is used to account for a known mismatch in the reference voltage of the actuation feedback circuitry. The coefficient for α_{x_ϕ} is written as:

$$\alpha_{\phi_j} = -\lambda_V^2 \frac{1}{m_{\text{TM}_j}} \frac{\frac{\partial^2 C_X^*}{\partial x^2} - 4 \frac{\left(\frac{\partial C_X^*}{\partial x}\right)^2}{C_{\text{total}}}}{\frac{\partial C_X^*}{\partial \phi}} \quad (8)$$

C_{total} gives the total capacitance between the test mass and the surrounding electrode housing [3]. Given the definition for ω_j^2 , it is now possible to define $\Delta\omega_{12}^2$ as being the difference in the individual stiffness of each test mass, and written as:

$$\begin{aligned} \Delta\omega_{12}^2 &= \omega_2^2 - \omega_1^2 \\ &= \omega_{2,0}^2 + \alpha_{x_2} F_{\text{max},2} + \alpha_{\phi_2} N_{\text{max},2} \\ &\quad - \left(\omega_{1,0}^2 + \alpha_{x_1} F_{\text{max},1} + \alpha_{\phi_1} N_{\text{max},1} \right) \end{aligned} \quad (9)$$

Equation (3) is now fully defined, however parameters depending on the orientation of the three bodies that comprise LPF must be taken into account. Therefore, it is necessary to include additional terms that take this effect into account, one such parameter is δ_{ifo} . The purpose of including δ_{ifo} is to capture the effect associated with the motion of the spacecraft with respect to the OMS optical bench [6]. Henceforth, δ_{ifo} can take on a different a value depending on the configuration of the system. Differences in δ_{ifo} include the re-alignment of the test masses, either done intentionally or by nature of the environment, and/or large fluctuations in temperature. It is then



necessary to write $\delta_{\text{ifo}} \equiv \delta_{\text{ifo}, k}$, where k refers to the different operational configurations of LPF. It is important to realize that $\delta_{\text{ifo}, k}$ describes a cross-coupling effect that affects the read-out oppose to a parameter that is a result from the dynamics of the three-body system. Regardless, it is necessary to include it in the fit in order to subtract all the signal induced, and perform goodness-of-fit tests to the resulting residuals. Therefore, equation (3) must be rewritten to include the additional $\delta_{\text{ifo}, k}$ parameter:

$$\begin{aligned} \Delta g_x(t) = & \Delta \ddot{x}(t) - \lambda_2 f_{x_2}(t) + \lambda_1 f_{x_1}(t) + \omega_2^2 \Delta x(t) \\ & + \Delta \omega_{12}^2 x_1(t) - C_1 \dot{f}_{x_1}(t) + C_2 \dot{f}_{x_2}(t) - \delta_{\text{ifo}, k} \ddot{x}_1(t) \end{aligned} \quad (10)$$

It was later determined that $k = \{1, 2, 3, 4\}$, which was done by assuming a fixed value for λ_j , C_j and $\omega_{j,0}^2$ across all experiments during the duration of the mission [2]. A summary of all the dynamical parameters used to define the LPF dynamics model is found in Table 1.

Parameters	
λ_1	α_{x_2} ($\text{kg}^{-1} \text{m}^{-1}$)
λ_2	α_{ϕ_2} ($\text{kg}^{-1} \text{m}^{-2} 10^3$)
$\omega_{1,0}^2$ (s^{-2})	$\delta_{\text{ifo},1}$
$\omega_{2,0}^2$ (s^{-2})	$\delta_{\text{ifo},2}$
C_1 (s)	$\delta_{\text{ifo},3}$
C_2 (s)	$\delta_{\text{ifo},4}$
α_{x_1} ($\text{kg}^{-1} \text{m}^{-1}$)	
α_{ϕ_1} ($\text{kg}^{-1} \text{m}^{-2} \times 10^3$)	

Table 1: Above is a table that includes a summary all of the parameters used to construct the dynamics model for LPF. As noted in the prior literature, coefficients $\delta_{\text{ifo},k}$ where $k = \{1, 2, 3, 4\}$ is classified as a cross-coupling of the read-out rather than a dynamical parameter; regardless, it must be taken into account to subtract the induced signal due to the tilt-to-length effect.

It is important to note that the dynamics described by the EOM in equation (10) give a description of the acceleration noise relative to the x-axis due to the motion of the test masses and spacecraft. However, in order to obtain a true estimate of the acceleration noise, denoted as $\Delta g(t)$, factors such as test mass misalignment



and rotational effects of the spacecraft must be taken into account. Therefore the complete estimate for acceleration noise is given by the following equation:

$$\Delta g(t) \equiv \Delta g_x(t) + \delta g_{\text{SC}}(t) + g_{\text{rot}}(t) \quad (11)$$

Where the $\delta g_{\text{SC}}(t)$ term is used to account for the misalignment between the test masses, the GRS and the spacecraft. This misalignment generates a cross-coupling effect that introduces cross talk signals in the output measurements of the IFO. It should be noted that the effect described by $\delta g_{\text{SC}}(t)$ is correlated to and influences the $\delta_{\text{ifo},k}$ parameter [6]. The $g_{\text{rot}}(t)$ term accounts for the inertial forces that act along the x-axis due to the rotation of LPF. Given that LPF is a rotating reference frame, rotational effects include the fictitious centrifugal force due to spacecraft angular velocity with respect to the J2000 reference frame, and Euler forces caused by a non-zero rotational acceleration.

3 Project Approach

As shown in the previous section, to estimate $\Delta g(t)$ would mean to find the parameters defined by LPF's dynamics model and correcting for rotational and misalignment factors. The former of which is done by conducting a series of system identification experiments used to measure the response of LPF and then fitting to a reference model in order to find the parameters as summarized by Table 1. Henceforth, the IRLS method was developed as an approach to perform the fit without any of the issues faced by previously established fitting techniques. The most important of which is being able to conduct the fit without prior knowledge of the noise in order to obtain an unbiased estimation for the noise environment. Furthermore, it was shown that when using a linear case for the dynamics, the IRLS method is equivalent to maximizing the likelihood via a Markov Chain Monte Carlo mapping of the Logarithmic Likelihood [5]. It is for this reason, that in order to support objectives (1) and (2) an in-depth understanding of the IRLS method is beneficial.



3.1 Iterative Reweighed Least Squares

As previously discussed, the IRLS method is used to fit the response of the LPF system, denoted as $\ddot{o}_{12}(n)$, to a linear model $g_m(n)$ defined by the linear combination $\sum_{i=1}^{N_{par}} \alpha_i x_i(n)$. In summary we have the following relationship, $\ddot{o}_{12}(n) = g_m(n) + r(n) = \sum_{i=1}^{N_{par}} \alpha_i x_i(n) + r(n)$, where the residuals capture any deviations of the data from the model. Moreover, we have to assume that the residuals are independent Gaussian random variables, and the time-series $x_i(n)$ are perfectly known. Under this assumption, to find the set of alpha parameters that best describes the data, or $\vec{\alpha}_{best}$, is simply a matter of minimizing the chi-square function. Here we begin with the definition of the chi-squared using the definitions for our data and reference model.

$$\chi^2 = \sum_{n=1}^{N_{data}} \frac{\left| \ddot{o}_{12}(n) - \sum_{i=1}^{N_{par}} \alpha_i x_i(n) \right|^2}{\sigma^2(n)} \quad (12)$$

However, as mentioned in [5], the noise that characterizes $\ddot{o}_{12}(n)$ is colored, causing our previous assumption to be nullified. A solution to this problem is to perform the fit in the frequency domain. Expressed in the frequency domain, the chi-squared is formally written as:

$$\chi^2 = \sum_{k=0}^{N_{data}} \frac{\left[\tilde{y}(k) - \sum_{i=1}^{N_{par}} \alpha_i \tilde{x}_i(k) \right] \left[\tilde{y}(k) - \sum_{j=1}^{N_{par}} \alpha_j \tilde{x}_j(k) \right]^*}{S(k)} \quad (13)$$

Where $\tilde{y}(k)$ is the Discrete Fourier Transform (DFT) of the time series $\ddot{o}_{12}(n)$, $\tilde{x}(k)$ is the DFT of the explanatory time series $\tilde{x}(n)$, j is considered to be a free variable and the asterisk represents the complex conjugate operation. The DFT of some time series $y(n)$ is defined by the following sum:

$$\tilde{y}(k) = \frac{1}{\sqrt{N_{data}}} \sum_{n=1}^{N_{data}} y(n) w(n) e^{-ink \frac{2\pi}{N_{data}}} \quad (14)$$

Where $w(n)$ is a window applied to the time-domain data to reduce spectral leak-

age, for the purposes of this project, the Blackman-Harris Window (BH96) is selected due to it is low spectral leakage combined with a reasonable bandwidth and amplitude error. To find the best estimate for the parameters in α is a matter of minimizing the chi-squared.

$$\begin{aligned} \frac{\partial \chi^2}{\partial \alpha_i} &= \sum_{k=0}^{N_{data}} \frac{\left[-\sum_{i=1}^{N_{par}} \tilde{x}_i(k) \right] \left[\tilde{y}(k) - \sum_{j=1}^{N_{par}} \alpha_j \tilde{x}_j(k) \right]^*}{S(k)} + \\ &\sum_{k=0}^{N_{data}} \frac{\left[\tilde{y}(k) - \sum_{i=1}^{N_{par}} \tilde{x}_i(k) \right] \left[-\sum_{j=1}^{N_{par}} \delta_{ij} \tilde{x}_j(k) \right]^*}{S(k)} = 0 \end{aligned} \quad (15)$$

Where δ_{ij} refers to the Kronecker-delta used to simplify the sum. By expanding the numerator of across each sum and simplifying, it is possible to rewrite the equation above as:

$$2 \sum_{k=0}^{N_{data}} \frac{Re\{\tilde{x}_j(k)\tilde{y}^*(k)\}}{S(k)} - 2 \sum_{i=1}^{N_{par}} \alpha_i \sum_{k=0}^{N_{data}} \frac{Re\{\tilde{x}_j(k)\tilde{x}_i^*(k)\}}{S(k)} = 0 \quad (16)$$

It is now possible to define a matrix \mathbf{A} and column vector \mathbf{B} such that:

$$\mathbf{A} = 2 \sum_{k=0}^{N_{data}} \frac{Re\{\tilde{x}_j(k)\tilde{x}_i^*(k)\}}{S(k)}, \mathbf{B} = 2 \sum_{k=0}^{N_{data}} \frac{Re\{\tilde{x}_j(k)\tilde{y}^*(k)\}}{S(k)} \quad (17)$$

Using the elements defined in equation (17), equation (16) can be expressed in matrix notation.

$$\mathbf{A}\vec{\alpha}_{best} = \mathbf{B} \quad (18)$$

To find $\vec{\alpha}_{best}$ becomes a matter of applying the left-hand inverse of matrix \mathbf{A} to the equation above.



A summary for implementing the IRLS algorithm is as follows:

1. Make an arbitrary guess for the noise PSD, i.e. Set all the frequency bins equal to 1.
2. Run the IRLS algorithm in the frequency domain by inverting equation (18) by substituting for A and B as described in equation (17).
3. Calculate the new noise PSD using Welch's method.
4. Repeat steps 2. and 3. until a desired threshold for $\bar{\alpha}$ is achieved.

4 Conclusion

During the project period objective (1), to understand and recreate the Iterative Reweighted Least Squares (IRLS) technique developed by the LISA collaboration, was successfully realized; along with it came a understanding of the underlying dynamics for the LPF satellite. These two key principles have laid the foundation in order to tackle objective (2) conducting a case study to investigate the impact of modifying the set of parameters specified to be solved for by the IRLS algorithm; the results of which could be used to better understand and account for cross-talk. Plans for the continuation of the project will be discussed with Dr. Daniele Vetrugno and the coordinators of the UF IREU program.



References

- [1] M. Armano, H. Audley, G. Auger, and J. Baird. Sub-femto-g free fall for space-based gravitational wave observatories: Lisa pathfinder results. *Physical Review Letters*, 116, 06 2016.
- [2] M. Armano, H. Audley, and J. Baird. Calibrating the system dynamics of lisa pathfinder. *Phys. Rev. D*, 97:122002, Jun 2018.
- [3] N. Brandt and W. Fichter. Revised electrostatic model of the lisa pathfinder inertial sensor. 2009.
- [4] G. Racca and P. Mcnamara. The lisa pathfinder mission. *Space Science Reviews*, 151:159–181, 2010.
- [5] S. Vitale, G. Congedo, and R. Dolesi. Data series subtraction with unknown and unmodeled background noise. *Phys. Rev. D*, 90:042003, Aug 2014.
- [6] G. Wanner and N. Karnesis. Preliminary results on the suppression of sensing cross-talk in LISA pathfinder. *Journal of Physics: Conference Series*, 840:012043, may 2017.