Thermal Conductivity Measurements of Sapphire Fibers

Allen Scheie*

Grove City College, 200 Campus dr., Grove City, PA, 16127†

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Abstract

In order to eventually utilize cryogenics in gravitational wave interferometers, research was carried out measuring the thermal conductivity of sapphire fibers as a candidate material for mirror suspension. Various thermal conductivity measurements were taken of two types of sapphire fibers at cryogenic temperatures. The results indicate that a thermo-optically polished composite fiber has a high peak thermal conductivity along the fiber of about 1.5×10^4W/m/K, meanwhile a grinded monolithic fiber of the same dimensions has an acceptable peak value of 1.0×10^4W/m/K. However, the thermal conductivity from the fiber head to the fiber rod was measured to be greater for a monolithic fiber than the composite fiber.

*Electronic address: `Scheieao1@gcc.edu`

†Ettore Majorana, Paola Puppo, University of Florida IREU Staff, and NSF

I. INTRODUCTION

Researchers hope to upgrade the next generation of gravitational wave detectors to have cryogenically cooled masses and mirrors. Cryogenically cooled components will significantly reduce thermal noise as well as thermal deformation, which will allow for much greater detector sensitivity at lower frequencies. However, the mirrors must be cooled without compromising the mechanical isolation from the suspension. One cannot simply connect a cryo-cooler to the masses, because the coolers add their own mechanical noise, as well as transmit all unwanted vibrations from the ground. Thus, heat must be extracted up from the mirrors through the suspension fibers. The challenge is finding a material that is suitably strong, has low mechanical loss, and has a sufficiently high thermal conductivity at low temperatures to keep the mirrors cool. Sapphire crystals are a good candidate material for this.

While the properties of bulk sapphire crystals are fairly well understood, the thermal and mechanical properties of sapphire fibers can vary depending on how the geometry and properties of the fiber. There are many different fabrication techniques, different fiber radii, and different polishing procedures. Each of these differences potentially has an effect on the thermal conductivity of the fiber. Before making the upgrade, it is necessary to understand how each of the proposed fiber materials behave at cryogenic temperatures. In the following experiments, we tested the thermal conductivity of two types of fibers. The first was a 1.6mm thermo-optically polished, two-head composite fiber (hereafter referred to as "1.6 mm TP 2HC" for "thermo-optically polished, two-head composite"), and the second was a grinded, two-head monolithic fiber (hereafter referred to as "1.6 mm G 2HM" for "thermo-optically polished, two-head composite"). The composite fiber had heads attached to a fiber rod using Alumina, while the monolithic fiber was grinded from a single crystal. We first measured the thermal conductivity of the fiber rods, and then we performed thermal conductivity tests on the heads of the fibers.

Figure 1: Composite Sapphire Fiber

Part I

Thermal Conductivity Measurements along the Fiber

II. SETUP AND EXPERIMENT

The experiments were carried out by mounting one of the sapphire fibers in a cryostat, placing a resistor on the top end of the fiber, putting two thermometer probes on the fiber itself at a given distance, and connecting the bottom of the fiber to the cryostat to act as a heat sink. We used a 30W 1kΩ resistor on top. The measured distance between the thermometer probes when we tested the 1.6mm TP 2HC fiber was 63.80 ± 0.05mm, and the measured distance between the probes when testing the 1.6mm G 2HM fiber was 65.00 ± 0.05mm. (For a more detailed description of the setup, see Appendix A.) Once the cryostat was sufficiently cool, we applied a voltage across the resistor, allowing it to heat the top end of the fiber. Heat would begin flowing down the fiber and into the heat sink, and a thermal gradient would be produced along the fiber. Eventually, the system would settle to stationary heat flow, and the thermal gradient would become constant over time. Once it reached stationary heat flow, we measured the temperature ready by each thermometer probe, the voltage across the resistor, the current through the resistor, and the average temperature of the thermometers. From this information, we could calculate the power to the resistor with the equation

$$P = IV \tag{1}$$

, and solve for the thermal conductivity of the rod at that temperature with the equation

$$P = \frac{A}{L}\kappa(T_1 - T_2) \tag{2}$$

, where A is the cross-sectional area, L is the length along the sample, κ is the thermal conductivity, and T_1 and T_2 are the temperatures read at the thermometer probes. Solved for κ, equation 2 equation becomes

$$\kappa = \frac{P\,L}{\Delta T\,A} \tag{3}$$

. Then, we re-set the resistor to a higher power, waited for the system to reach stationary heat flow again, and repeated the measurements. Using this procedure, we gathered data for various temperature differences along the fiber.

We first took thermal conductivity measurements along the 1.6mm TP 2HC fiber, and then we repeated the measurements with the 1.6mm G 2HM fiber.

III. RESULTS

A. Thermo-optically polished, Two-head Composite

The results indicate that the 1.6mm TP 2HC fiber has a relatively high thermal conductivity, especially around the peak value at 30K of roughly 1.5×10^4W/m/K. Figure 2 shows the measured values of thermal conductivity plotted against the average temperature of the thermometers. The relationship is fairly linear until around 20-30 K, when the thermal conductivity slopes off, and then decreases with increasing temperature. This peak effect of thermal conductivity is likely due to phonon scattering at higher temperatures. For a more complete discussion of thermal conductivity along Sapphire fibers, see Hall et. al. [1]. Figure 2 also shows the uncertainty bars for the different values. (For a discussion of how this uncertainty was calculated, see Appendix C.) The data only goes down to 7.85K due to the fact that we used a pulse-tube cryostat which is limited in temperature range.

B. Grinded, Two-head Monolithic

The results for the 1.6mm G 2HM fiber indicate that this fiber does not have as high of a thermal conductivity as the 1.6mm TP 2HC fiber, especially around the peak value at 30K.

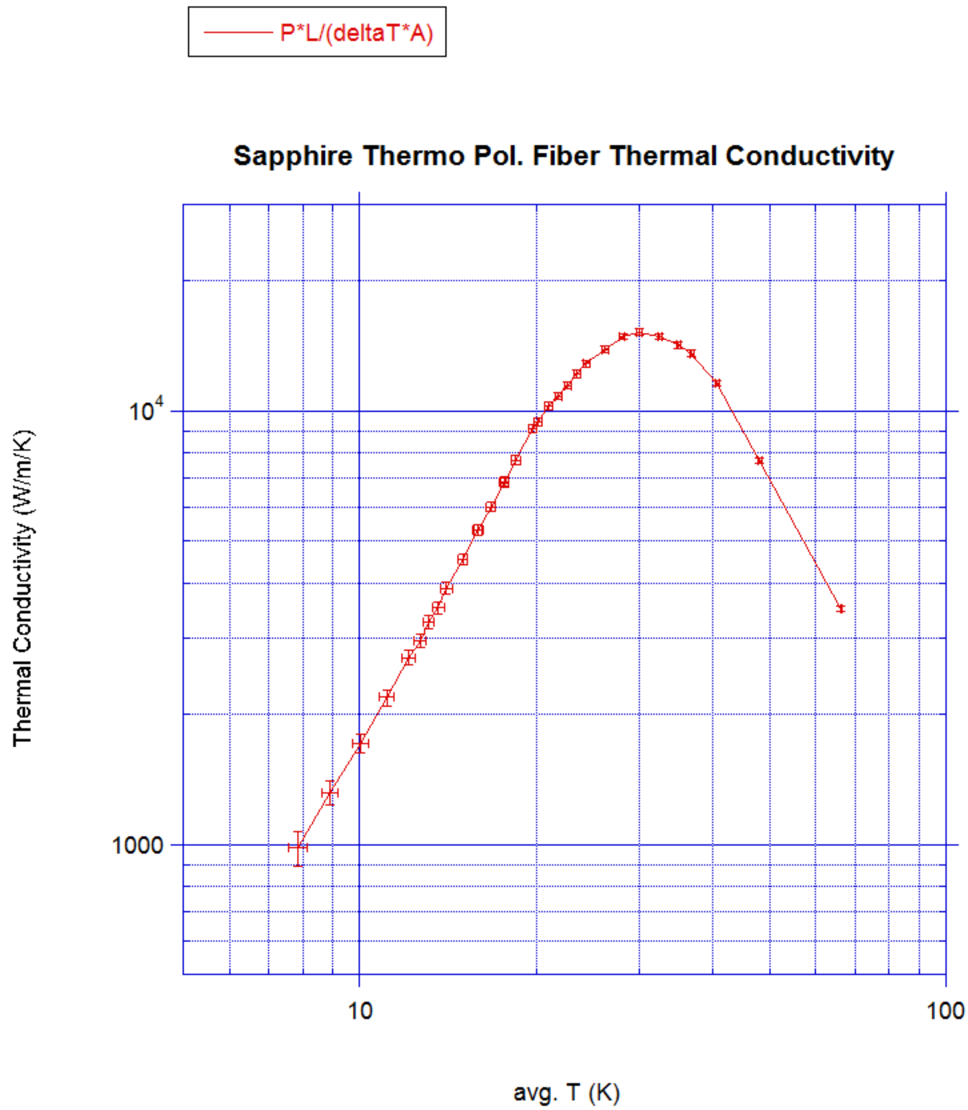

Figure 2: Thermal Conductivity vs. Average Temperature, TP 2HC

Figure 3 shows the measured values of thermal conductivity plotted against the average temperature of the thermometers, with the uncertainty bars for the different values. (For a discussion of how this uncertainty was calculated, see Appendix C.) The shape of the curve is similar to that of the 1.6mm TP 2HC fiber, but with lower thermal conductivity values, and with the peak κ value of 1.0×10^4W/m/K at a slightly higher temperature of 34.3K.

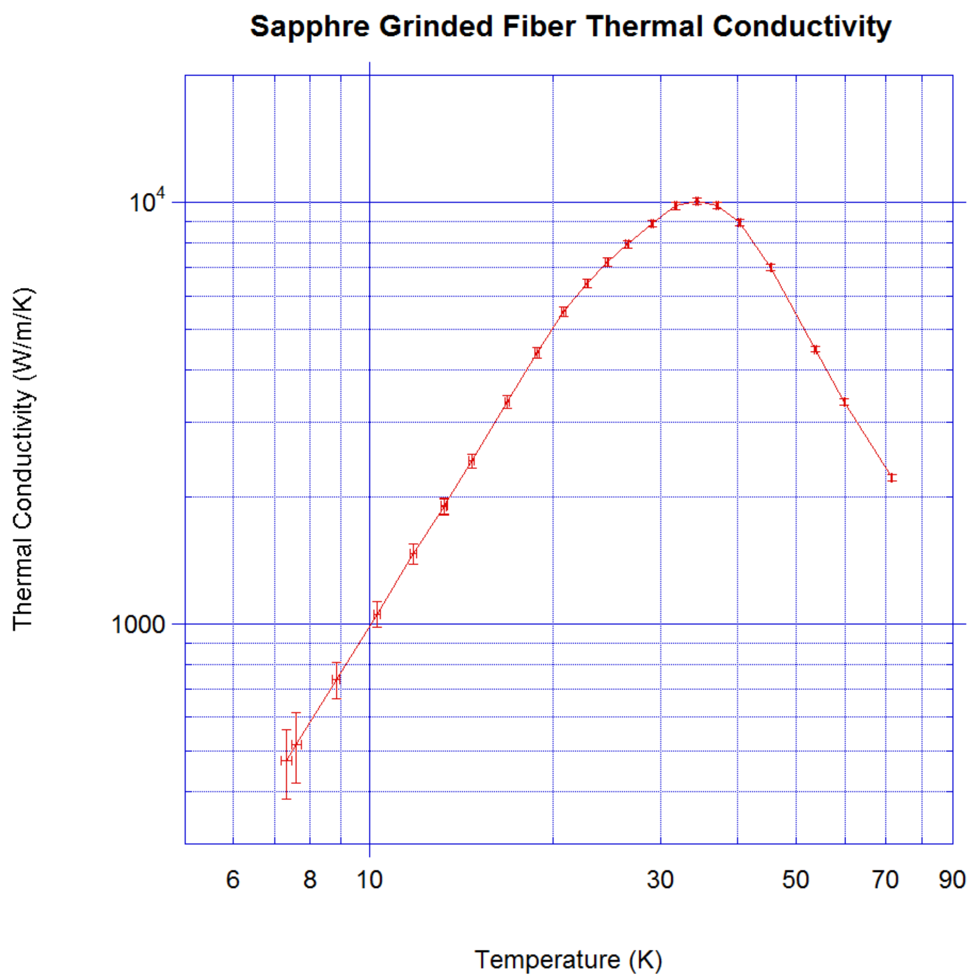

Figure 3: Thermal Conductivity vs. Average Temperature, G 2HM

C. Comparison to Other Fibers

Figure 4 shows the data from the 1.6mm TP 2HC and 1.6mm G 2HM fibers plotted with measurements from other types of sapphire fibers. (It should be noted that the data in this graph from the other samples is not exact; they are there merely for the sake of comparison. This data was taken from [2, p. 16].) Notice that in the temperature range from 30 to 40 K, the thermal conductivity of the 1.6mm TP 2HC fiber is higher than the values for the other types of fibers, and is closest to the values for bulk sapphire. For this temperature range, this type of fiber would likely be an excellent candidate for mirror suspension. Meanwhile, the 1.6mm G 2HM fiber is closer to the other values for thermal conductivity, indicating an acceptable, but not exceptional candidate for suspension fiber. For the data points in the range beyond 40 Kelvin, the slopes of the 1.6mm TP 2HC and

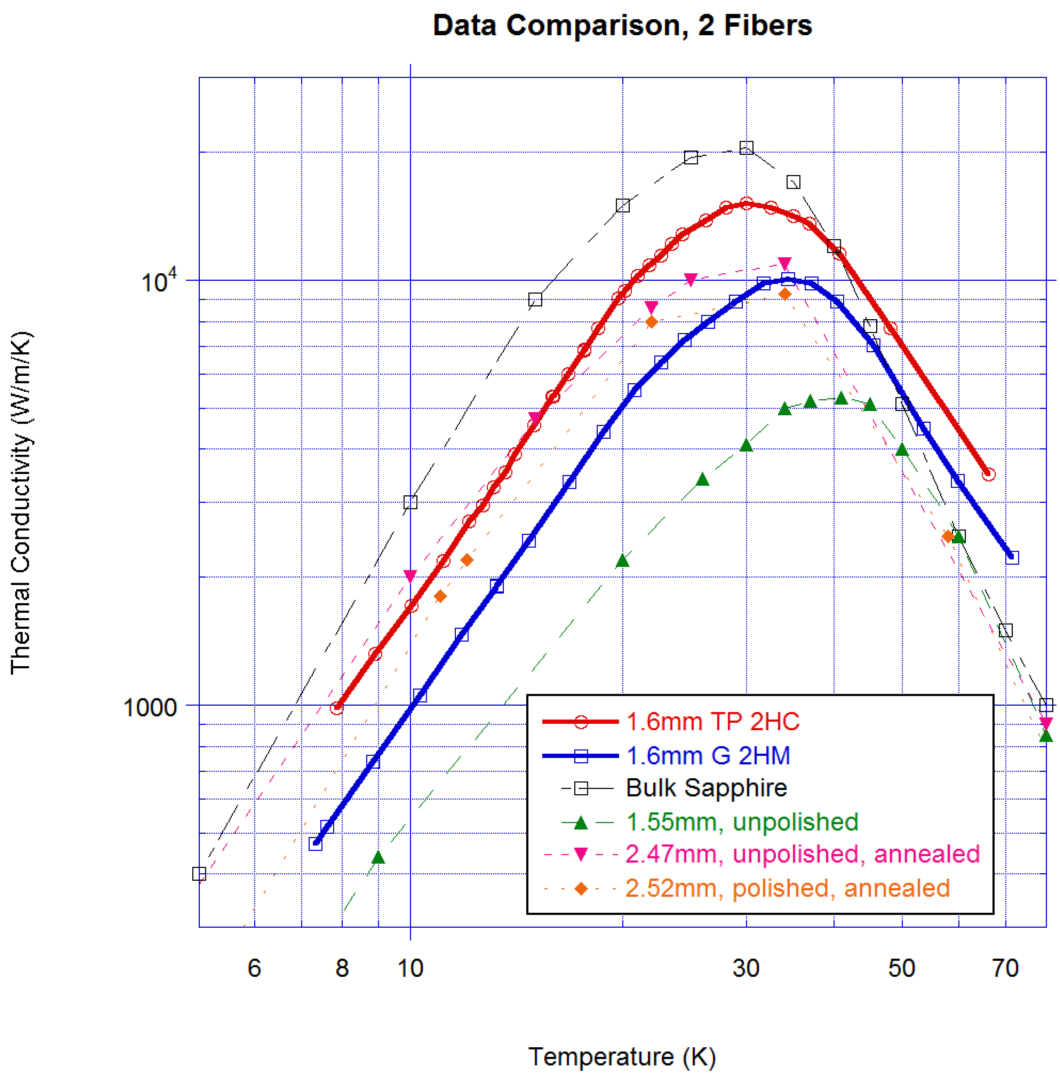

Figure 4: Comparison of Thermal Conductivity Measurements

1.6mm G 2HM plots are different than all other sapphire measurements. It is possible that this is due to radiative power loss from a hot resistor (one of the resistors did begin melting during the experiment), or it could be due to the properties of the cryostat outside the range of very low temperatures. This will be a topic for further investigation.

Part II

Thermal Conductivity Measurements from Head to Rod

After measuring thermal conductivity along the fibers, we attempted to measure the thermal conductivity from the head at the end of the fiber to the rod of the fiber itself. This information is of interest because the suspension fibers will eventually be attached to the attenuator with of a similar type of head, so it is important to understand how heat flows through the different types of heads to the fibers.

IV. HEAD MEASUREMENTS WITH ALUMINUM RESISTOR HOLDER

The first measurement we took was the temperature difference between the outside of the Aluminum fiber holder and the rod of the 1.6mm G 2HM fiber. Ideally, the aluminum head causes no extra power loss, so it was initially assumed that the aluminum would take on the same temperature as the fiber head when the system reaches stationary heat flow. Working under this assumption, we mounted two thermometers onto the 1.6mm G 2HM fiber; one clamped directly to the side of the aluminum resistor holder, and one on the copper block mount on the fiber below the aluminum head. We then repeated the measurements for stationary heat flow. However, we discovered that the temperature difference between the aluminum head and the fiber itself reversed as temperature increased. Below 28K, the fiber was at a higher temperature than the aluminum head, and above 28K, the head was at a higher temperature. Clearly, thermal conductivity cannot be calculated from this data. ΔT changes signs, which means that thermal conductivity would be negative when ΔT is negative (which is impossible and meaningless), and thermal conductivity would be infinite when ΔT is zero (which is not the case for sapphire of any kind).

We attempted to re-do the experiment by drilling a hole in the aluminum head and mounting the thermometer directly against the sapphire fiber head. However, we saw the same result: at low temperatures the thermometer on the fiber was reading a higher temperature than the thermometer on the fiber head, and then ΔT changed signs above about

25K. Thermal conductivity could not be calculated from this data either; we needed a new way to measure heat flow through the head. For a more detailed description of this setup and the data from the experiment, see Appendix B.

V. HEAD MEASUREMENT WITH TEFLON RESISTOR HOLDER

A. Setup

Due to the problems encountered with using the aluminum head to measure thermal conductivity, we decided to design a different way to hold the resistor on the top of the fiber which would involve less thermal contact to various parts of the fiber head. The new design was two rectangular pieces of Teflon clamping the resistor to the fiber by means of Teflon bolts, and it is depicted in Figure 5. The thermometer was held to the fiber head by means of a plastic zip-tie. In these experiments the small copper ring in-between the resistor and the fiber (see Appendix A) was replaced by a brass washer; the ring was getting deformed, and it was questionable whether it was thick enough to effectively protect the top of the composite fiber. We tested the setup in liquid nitrogen to see if it was robust at low temperatures, and no cracks or deformations were noticed.

For both fibers in this experiment, the top of the copper block thermometer mount was 6.25±0.05mm below the bottom edge of the fiber head. Due to the second thermometer being underneath a zip-tie it was difficult to precisely determine the position the thermometer. However, each time we tried to place it at exactly the middle of the fiber head. Thus, we estimated that the position of the head thermometer was 2.5 ± 0.7mm above the bottom edge of the fiber.

The $P = 0$ equilibrium ΔT for the 1.6mm G 2HM fiber was 0.183K, and the $P = 0$ equilibrium ΔT for the 1.6mm TP 2HC fiber was 0.658K. We subtracted from the respective ΔT measurements as a calibration difference.

We repeated the measurements for stationary heat flow, first for the 1.6mm G 2HM fiber, and then again for the 1.6mm TP 2HC fiber.

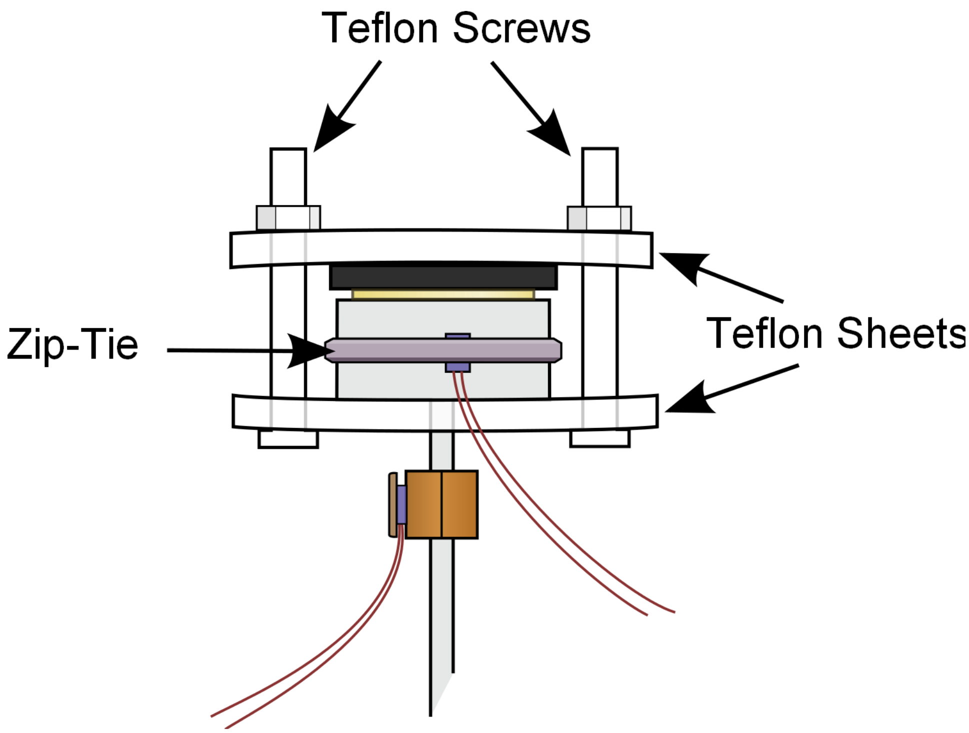

Figure 5: Teflon Resistor Mount

B. Results

The results from these tests are far more encouraging. The temperature difference was positive for the entire set of measurements for both types for fibers.

To calculate thermal conductivity, we had to create a model with which to estimate the constants in Eq. 2. The geometry of the head connected to the rod is more complex, and the characteristic heat flow is not trivial. However, we came up with a workable model by making some simplifying assumptions. First, we assumed that the head of the fiber has uniform cross-sectional temperature along the axis of the fiber. Secondly, we assumed that all power from the large radius portion (the head) flows directly into the small radius portion (the rod). Thus, we could assume

$$\triangle T = \triangle T_h + \triangle T_r$$

and

$$P = P_h = P_r$$

where $\triangle T$ is the temperature difference between the two thermometers, $\triangle T_h$ is the temperature difference between the top thermometer and the bottom of the head, $\triangle T_r$ is the temperature difference between the bottom of the head and the thermometer on the fiber rod, P is the power provided by the resistor, P_h is the power provided to the cross-section of the head where the top thermometer was mounted, and P_r is the power provided to the top of the fiber rod from the head.

Using our model and Eq. 2, we arrived at the following equations

$$\Delta T_h = \frac{P_h\, L_h}{\kappa\, A_h} \quad \text{and} \quad \Delta T_r = \frac{P_r\, L_r}{\kappa\, A_r}$$

where L_h is the length along the head between the top thermometer and the bottom of the head (in both cases 2.5 mm), L_r is the length along the rod from the head to the bottom thermometer (in both cases 6.25 mm), A_h is the cross-sectional area of the head ($7.85 \times 10^{-5}\text{m}^2$) and A_r is the cross sectional area of the rod ($2.01 \times 10^{-6}\text{m}^2$). Using the assumptions listed above, these equations could be solved to

$$\kappa = \frac{P}{\Delta T}\left(\frac{L_h}{A_h} + \frac{L_r}{A_r}\right) \tag{4}$$

. For both cases, $\frac{L_h}{A_h} + \frac{L_r}{A_r}$ is constant and equal to 3140m.

Using this simplistic model, we quantitatively calculated the thermal conductivity from the head measurements from both fibers, as is shown in Figures 6 and 7. The resulting curve has the same general behavior as the previous data in Figure 4. However, there are several things worth noting about this data.

First, the peak thermal conductivity from the head to the fiber is higher for the monolithic fiber than the composite fiber, which is the opposite of the thermal conductivity measured along the fibers. This is possibly due to an increased thermal resistance in the interface between the connected head and the fiber in the composite sample. Secondly, the thermal conductivity in these measurements is roughly a factor of two lower than the measurements taken along the fiber. This could be due to an overly-simplistic model, a systematic error in the experiment, or perhaps some other unaccounted for effect. This will have to be a topic for future investigation. Thirdly, there is a small bump in both graphs to the left of the peak. This is also an unexplained phenomenon, though it was reproducible. It is unknown what the cause of this is. Clearly, there is much to explain in the interface between the sapphire head and the sapphire rod.

One final note is that the low temperature data in Figure 7 do not match those in Figure 6. This is probably due to the fact that when taking the measurements of the monolithic fiber,

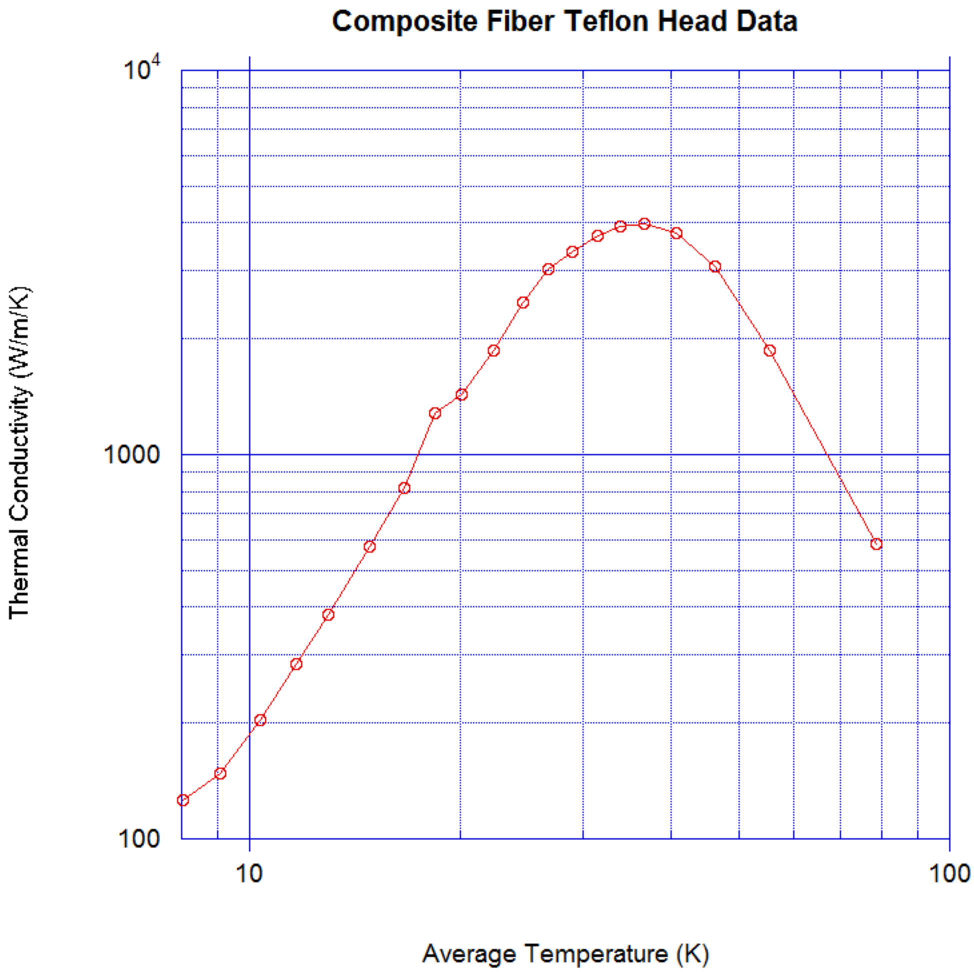

Figure 6: Data from Composite Fiber

the cryo-cooler was not yet at true equilibrium and was very slowly dropping in temperature. If this is the case, then the first few data points from both graphs can be disregarded.

VI. CONCLUSION

The thermal conductivity measurements of the 1.6mm TP 2HC fiber indicate that this type of fiber has very high thermal conductivity around the peak range of 20-40 K, and it is likely a good candidate to use for mirror suspension. The thermal conductivity measurements of the 1.6mm G 2HM fiber indicate an acceptable value of thermal conductivity around the peak range of 20-40 K. The thermal conductivity from the fiber head to the fiber rod needs to be studied in greater detail, but the initial results indicate that the thermal

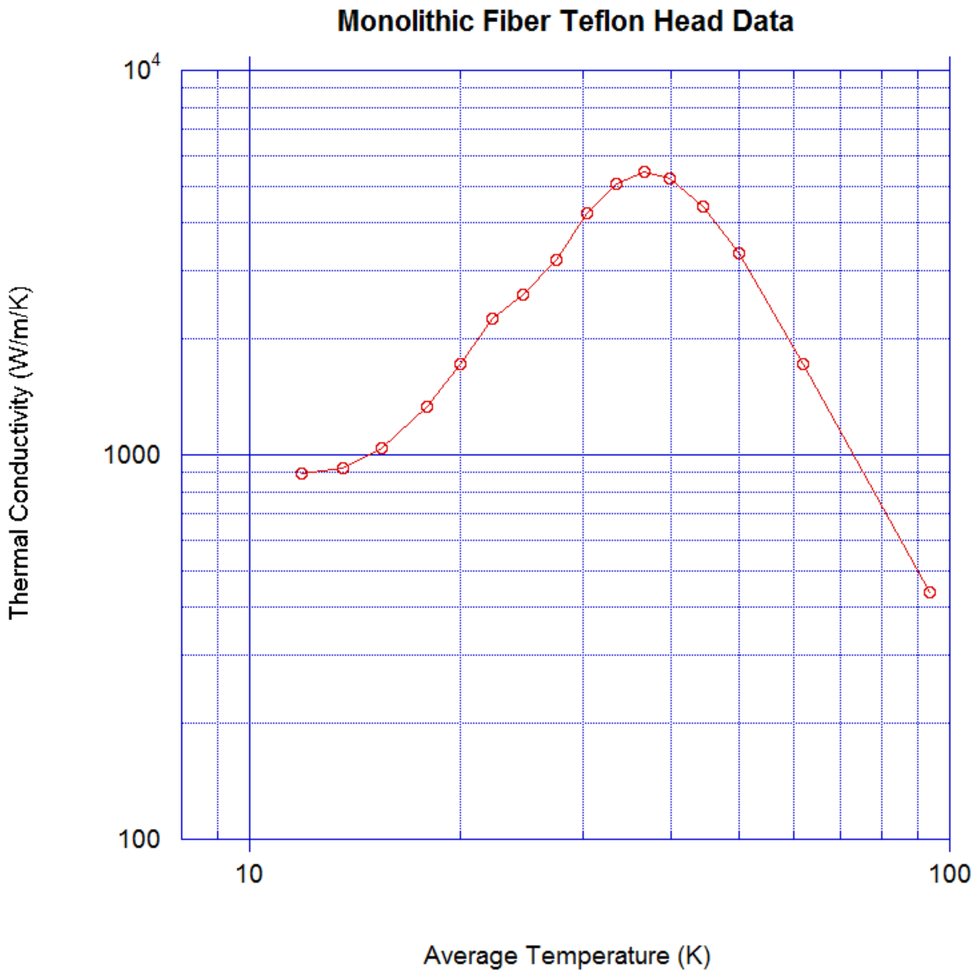

Figure 7: Data from Monolithic Fiber

conductivity is higher for a monolithic sample than it is for the composite sample.

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[1] M. Hall, N. DiNardo, R. DeSalvo, T. Tomaru, F. Fidecaro, "Surface Roughness and Thermal Conductivity for Sapphire Fibers: Reducing the Thermal Noise in Gravitational Wave Interferometers." LIGO-T030126-00-D (May 29, 2003).

[2] G. Hofmann, C. Schwarz, R. Douglas, Y. Sakakibara, J. Komma, D. Heinert, P. Seidel, A. Tünnermann, S. Rowan, K. Yamamoto, E. Majorana, and R. Nawrodt, "Mechanical Loss and Thermal Conductivity of Materials for KAGRA and ET", ELiTES Workshop (April 19th 2013).

Appendix A: FIBER THERMAL CONDUCTIVITY EXPERIMENT SETUP AND DETAILS

1. Fiber Dimensions

The two fibers that we used (1.6mm TP 2HC and 1.6mm G 2HM) were of the same dimensions. From the end of the head to the end of the head, the fibers measured 100.00 ± 0.05mm. The thickness of each head was 5.30 ± 0.05mm, and the diameter of each head was 10mm. The diameter of the fiber rod itself was 1.6mm. The uncertainty for fiber and head radius is discussed in Appendix C.

2. Experiment Setup

The following method was used for both types of fibers when measuring thermal conductivity along the fiber.

First, we put the thermometer mounts onto the sapphire fiber by clamping the fiber with two copper blocks near each end, and screwing a small, gently bent copper sheet onto the block which would hold the thermometer in place (for a diagram of how the thermometer was mounted, see Figure 8). Next, we mounted the fiber into its holder, which is a large copper piece which attaches to the cryostat. After that, we placed a 30W 1kΩ resistor on the top of the fiber by means of a specially designed aluminum holder. (When mounting the fibers, we attached the resistor holder first and the mount second, in an attempt to put less unnecessary stress on the fiber.) Due to the fiber slightly protruding from the end of the fiber head of the TP 2MC fiber, it was necessary to place a small copper ring in between the resistor and the fiber itself in order to provide good thermal contact. (For the sake of consistency, this small ring was also used for the grinded monolithic fiber.) We then mounted the fiber in the cryostat, and placed the thermometers in their mounts by tightening the screws to the copper sheets. For a picture of the completed setup, see Figure 9. The measured distance between the copper blocks holding the thermometers was 63.80 ± 0.05mm for the TP 2HC, and 65.00 ± 0.05mm for the G 2HM. Because copper has a much higher thermal conductivity than Sapphire, we assumed that the copper was at a uniform temperature compared to the thermal gradient along the fiber, and the distance between the copper blocks could be taken

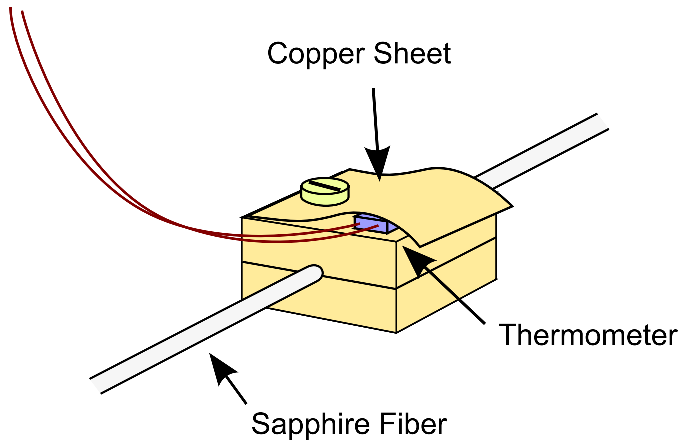


Figure 8: Thermometer Mount

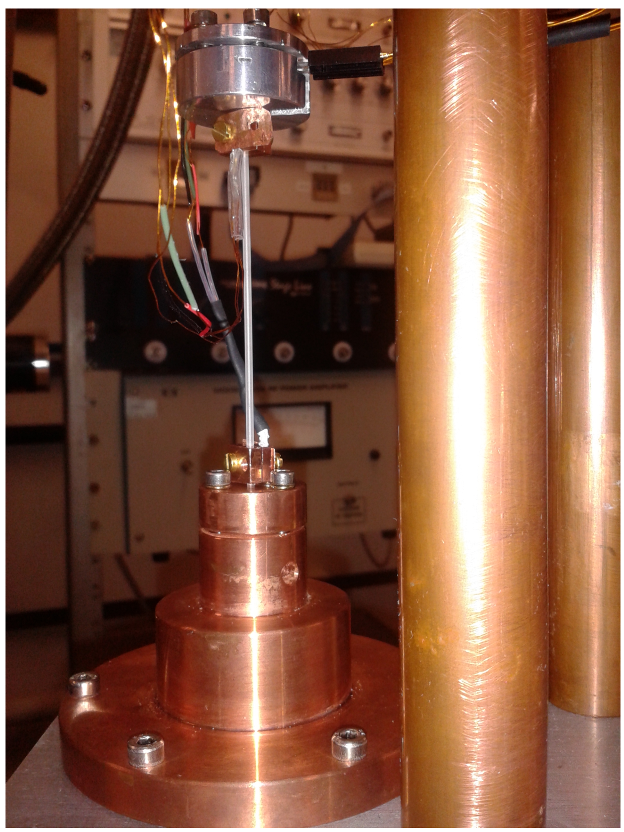


Figure 9: Completed Setup

to be the distance between the thermometer probes.

While running the experiment itself, the thermometer readouts and the voltage across the resistor were recorded in 30 second increments by a LabView program. To measure voltage, we used an Agilent Technologies 3458A multimeter which had computer control capability.

To measure current, we used an HP 3468A multimeter which did not have computer interface capabilities, so we manually recorded the current each time the setup reached stationary heat flow. 32 thermal gradient equilibria were investigated over the course of six days for the 1.6mm TP 2HC fiber, and 23 thermal gradient equilibria were investigated over four days for the 1.6mm G 2HM fiber. Each time the resistor was turned off, the fiber would slowly cool to equilibrium at roughly 6K. The data for the 1.6mm TP 2HC fiber was taken in two sets, with the time value re-starting to zero in-between. The data for the 1.6mm G 2HM fiber was taken in one set, with the time value set to zero at the beginning.

Appendix B: HEAD MEASUREMENTS USING ALUMINUM MOUNT

1. Aluminum Head Measurement

a. Setup

The first measurement we took of the fiber head was the temperature difference between the outside of the aluminum resistor holder and the rod of the 1.6mm G 2HM fiber. We mounted two thermometers onto the 1.6mm G 2HM fiber; one clamped directly to the side of the aluminum resistor holder at the top of the fiber with a small copper sheet, and one on the copper block mount 1.40 ± 0.05mm below the aluminum head. In this experiment, we inserted a small copper ring in-between the resistor and the sapphire fiber head. The setup is shown in Figure 10. The fiber was placed in the cryostat, and various voltages were applied to the resistor at the top of the fiber. After letting the system reach steady heat flow, we recorded the temperatures of each thermometer, and the voltage and current through the resistor. The $P = 0$ equilibrium ΔT was 0.9K, which we subtracted from all the ΔT measurements.

b. Results

The results are surprising. First, the temperature difference between the fiber head and the fiber itself reversed as temperature increased. Figure 11 shows this effect, and how the temperature of the fiber is higher than that of the temperature of the head until about 28K, at which point the temperature of the aluminum head is higher than that of the fiber. An

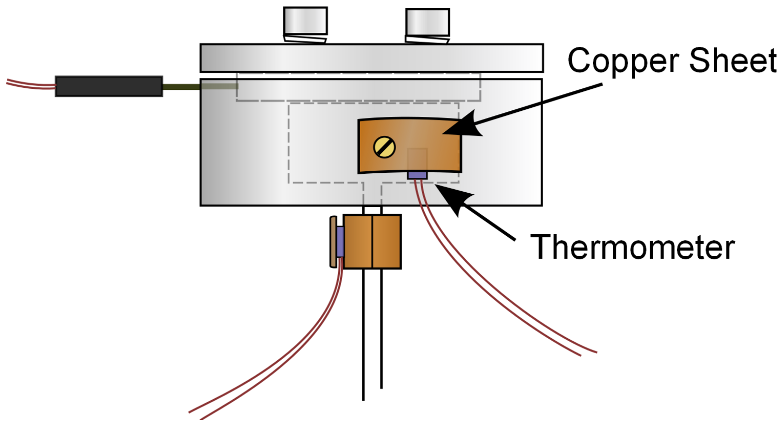


Figure 10: Thermometer Mount on Outside of Aluminum Head

additional interesting characteristic of the data is that at an average temperature of 25K (when power to the resistor is 0.8 W), the temperature difference spikes downward. This effect was reproduced on two separate days of the data run, so it is probably due to some physical phenomenon. The cause is unknown; this may be a topic for further investigation. This may reveal something very interesting about the nature of the sapphire fibers.

It should be noted that the "negative gradient" never occurred in the raw data. This was a result of subtracting the calibration difference from the original ΔT measurement, which was obtained by assuming that when the system is at thermal equilibrium at $P = 0$, the two thermometers are at the same temperature. Thus, the $P = 0$ equilibrium ΔT (0.9K in this case) was treated as the calibration difference of the two thermometers. If these assumptions are flawed, the negative gradient may or may not have been real.

Clearly, thermal conductivity cannot be calculated from this data. Thermal conductivity would be negative when ΔT is negative, and thermal conductivity would be infinite when ΔT is zero.

2. Sapphire Head Measurement

a. Setup

Because of the difficulties encountered in getting thermal conductivity out of the measurements described above, we decided to mount the thermometer directly onto the head of the 1.6mm G 2HM sapphire fiber, in the hopes of getting a more direct measurement of the

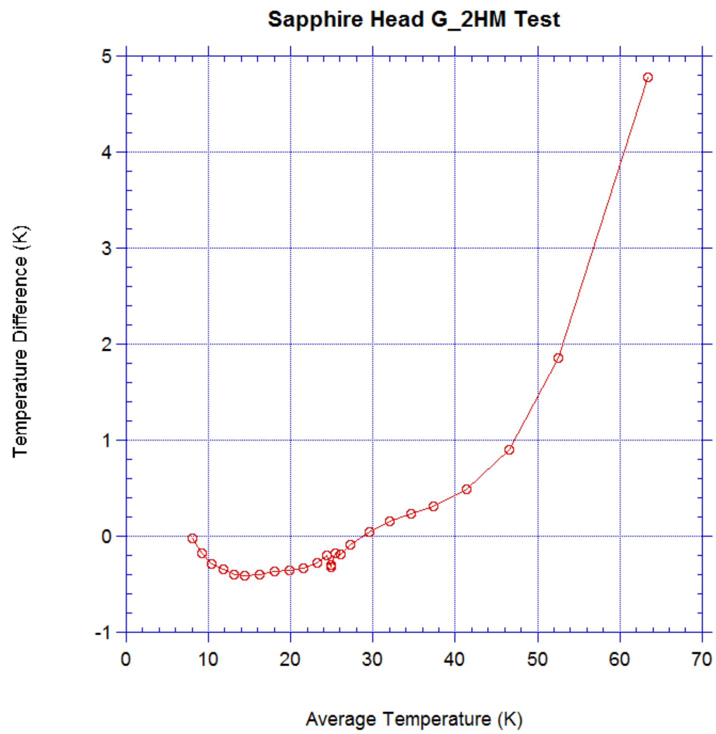


Figure 11: ΔT vs. T for aluminum head mount

temperature gradient.

The setup was as follows: We drilled a hole in the bottom of the aluminum mount which opened up on the fiber head cavity. Next, we drilled a threaded hole in the side of the aluminum mount that met the hole drilled from the bottom. We inserted a thermometer in the bottom hole, and a Teflon screw in the side hole. The screw pressed the thermometer directly up against the sapphire fiber, giving a direct thermal contact to the fiber head. (See Figure 12.) The other thermometer was mounted on a copper block clamped to the fiber rod 1.10 ± 0.05mm below the aluminum head. This time, we used a brass washer between the resistor and the fiber instead of the small copper ring we had previously used. We then repeated the measurements for stationary heat flow. The $P = 0$ equilibrium ΔT was 0.256K, which we subtracted from all the ΔT measurements.

b. Results

As is shown in Figure 13a, the temperature difference still goes negative between 8K and 25K. This would seem to indicate that the edge of the fiber is at a lower temperature than

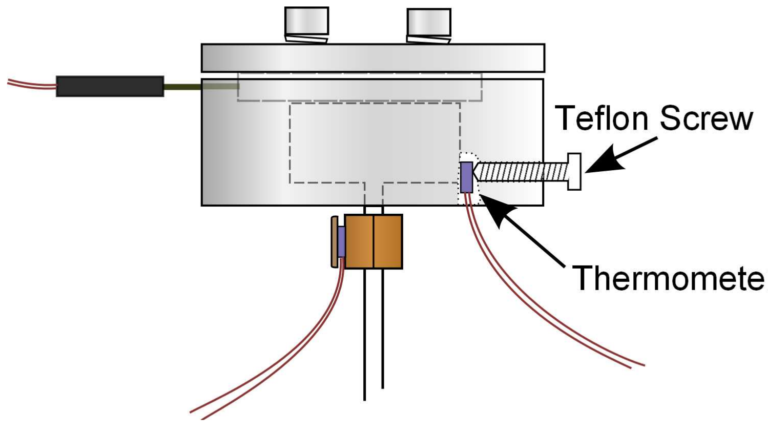


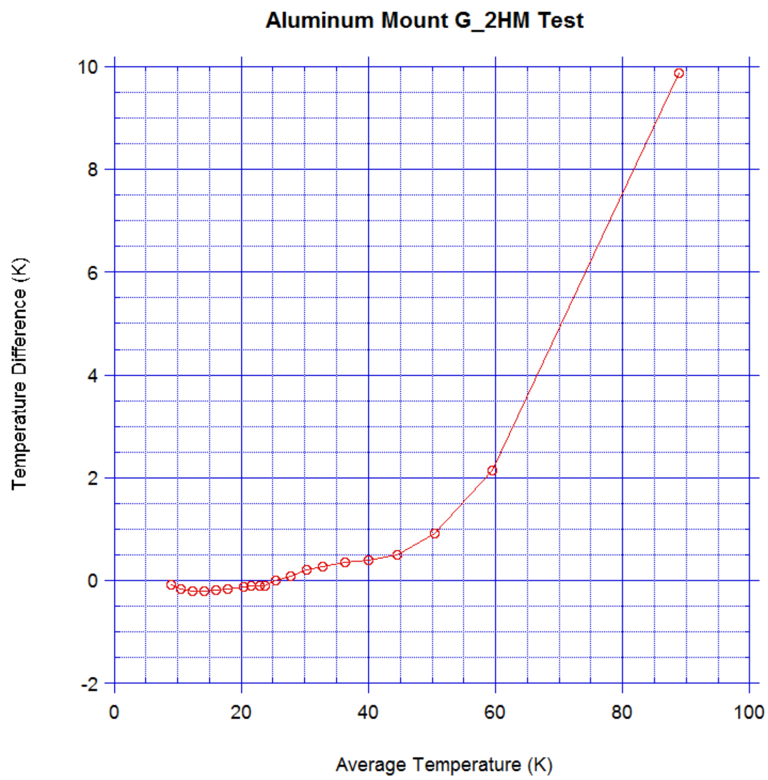
Figure 12: Thermometer Mount Inside Aluminum Head

the fiber directly below it. Figure 13b shows the data on the same axes as Figure 11 for the sake of comparison. While the ΔT does not go as far negative in the measurements, it still follows the same pattern as the data taken from the outside of the Aluminum head. For the reasons mentioned above, thermal conductivity calculations are not possible from such data.

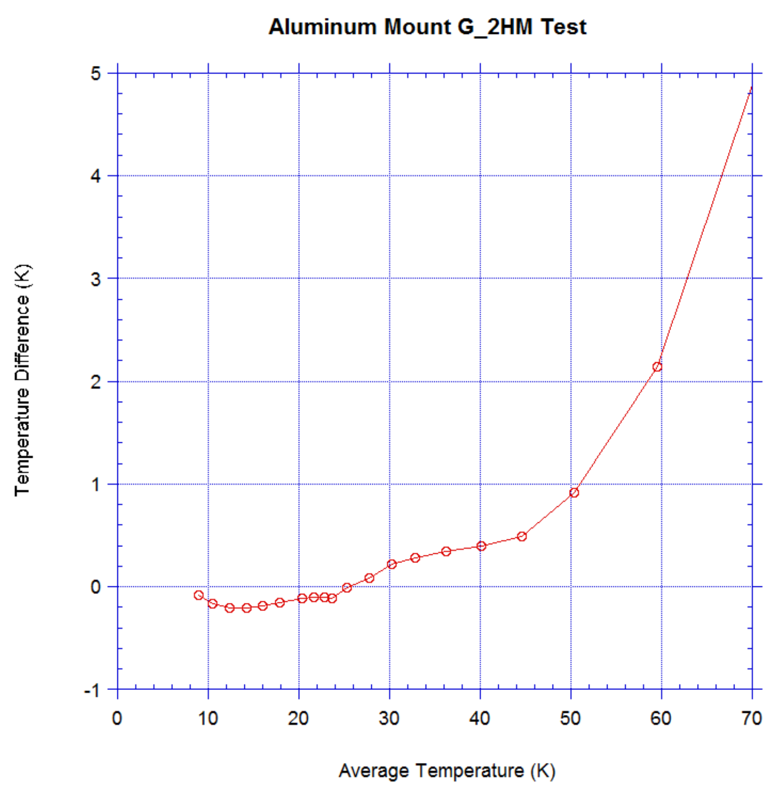
c. Interpretation

These results are unexpected and difficult to interpret. It could be that these measurements are correct, and the edge of the fiber head is really at a lower temperature than the fiber rod below the head. Contact with the aluminum head could be causing some sort of heat loss, and thus the parts of the fiber next to aluminum are at a lower temperature; or the results could be due to some other unaccounted for physical effect. However, it may be that the measurement is faulty, and the thermometer is being affected more by the aluminum head, which has the odd behavior of switching direction of the thermal gradient.

Finally, the thermal gradient direction change could be due to the "calibration difference" being a real physical temperature difference. In that case, it would be incorrect to treat the $P = 0$ equilibrium ΔT (0.256K in this case) as a difference in calibration. This is a plausible explanation for two reasons. The first reason is that the "calibration difference" varies widely between different measurement sessions. For example, the $P = 0$ equilibrium ΔT in the experiment with the thermometer inside the aluminum head is only about 28% of the $P = 0$ equilibrium ΔT from the experiment with the thermometer outside the aluminum head. This



(a) ΔT vs. T for fiber head mount



(b) ΔT vs. T , close up

Figure 13: ΔT vs. T for sapphire head mount

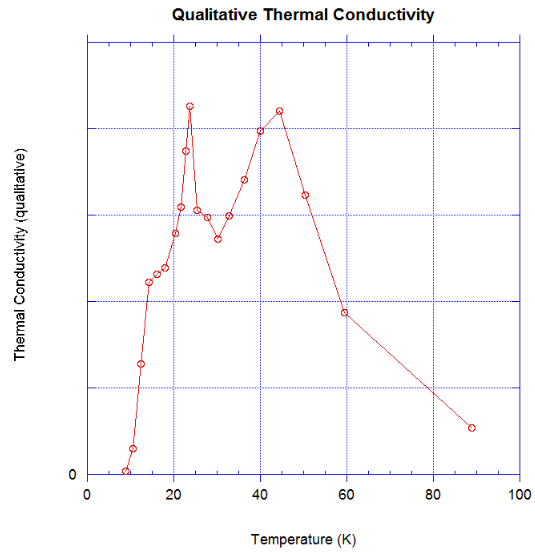


Figure 14: Qualitative Thermal Conductivity vs. Temperature

wide variation indicates something other than just thermometer calibration, because one would assume that is fairly constant. The second reason is that the temperature difference in the raw data from the above measurements never goes negative; the raw gradient just decreases over a given temperature range. This indicates the possibility that the "negative gradient" is just a product of an invalid correction.

Of course, if this last hypothesis is true, and the negative gradient is not real, one must still account for why the gradient decreases over the temperature range, to reach a minimum around 18K. Figure 14 shows a plot of a qualitative thermal conductivity calculation ($P/\Delta T_{raw}$) plotted against temperature. According to other measurements of sapphire (including our own), sapphire thermal conductivity should reach a maximum somewhere between the two peaks on the graph; this data is very strange when compared to the existing literature on sapphire conductivity [2].

Appendix C: UNCERTAINTY ANALYSIS FOR THERMAL CONDUCTIVITY ALONG FIBER

Subsection 1 lists and explains the uncertainties for the measurements we took in the experiments. Subsection 2 describes generally how I propagated uncertainty in my calculations for the thermal conductivity along the sapphire rods. Unfortunately, there was not enough time to do a full uncertainty analysis of the thermal conductivity measurements of the data from the heads of the fibers.

1. Uncertainties in Measured Values

a. Fiber Length

The standard uncertainty for the distance between the thermometer probes was 0.02mm. This value for uncertainty is based on the smallest measuring increment of 0.05mm on the calipers, which has a triangular distribution with a full width equal to twice the smallest measuring increment. For a triangular distribution having a full width $2a$, the standard uncertainty u is

$$u = \frac{a}{\sqrt{6}}. \tag{C1}$$

which yields a value of 0.02mm for standard uncertainty.

b. Cross-sectional Area

For the TP 2HC fiber, this value of uncertainty was not taken into account in my analysis, due to the very precise polishing technique applied to the fiber. The variations in radius should be on the order of less than a micron, which is negligible when compared to the other sources of uncertainty in the experiment.

However, this contribution was taken into account for the G 2HM fiber, due to the rougher surface. According to the documentation provided, the grinded surface should have a peak-to-peak variation of 3.28 microns. I propagated this through the equation for area using Eq. C4, and got a value of $\pm 8 \times 10^{-9}\text{m}^2$ for raw uncertainty. To get the standard deviation value from this, I applied a uniform distribution to the uncertainty. For a uniform distribution having a full width $2a$, the standard uncertainty u is

$$u = \frac{a}{\sqrt{3}}. \tag{C2}$$

, which yielded a value of $\pm 5 \times 10^{-9}\text{m}^2$.

c. Current

The standard uncertainty for the current was $\pm$(1% of reading + 30 counts). This was taken from the measurement accuracy specifications on page 1-3 of the manual for the HP 3468A multimeter, which was used to measure current.

d. Voltage

The standard uncertainty for voltage was $\pm$(14ppm of reading + 0.00003V). This was taken from the measurement accuracy specifications on page 284 of the manual for the Agilent 3458A multimeter, which was used to measure voltage, taking into account the fact that most of our measurements fell within the 10-100 V range.

e. Temperature Difference

The standard uncertainty for temperature difference was different for the two fibers. For the TP 2HC fiber, it was usually 0.009K, though it was higher for high average temperature readings. For the G 2HM fiber, it was usually 0.006K, though higher for high average temperature readings. These values for uncertainty are based off the calibration difference between the two thermometers. When the system would cool to equilibrium when resistor power was zero (for example, when the cryostat was left on over night), the two thermometers would never read exactly the same temperature. We let the system cool to equilibrium six times over the course of the TP 2HC experiment, and four times over the course of the G 2HM experiment. The temperature differences at $P = 0$ equilibrium are recorded in Table I and Table II respectively.

For the TP 2HC data (Table I), the average of these temperature differences is 0.572K, and the difference between the maximum and minimum values is 0.03K. To increase the accuracy of the temperature difference data, I subtracted the average value 0.572K from each measured value of ΔT. Given that the actual calibration difference between the thermometers at any given time could be anywhere in the range in Table I, I applied uniform distribution with a full width equal to 0.03K to calculate standard uncertainty for ΔT (see Eq. C2), which yields a value of 0.009 for low temperatures.

For the G 2HM data (Table II), the average of these temperature differences is 0.272K, and

Table I: Thermal Equilibrium ΔT Values, TP 2HC

Data Set	Time	ΔT
1	51120	0.57
	84510	0.56
	160920	0.56
2	86640	0.57
	179550	0.58
	257220	0.59

Table II: Thermal Equilibrium ΔT Values, G 2HM

Time	ΔT
0	0.259
71130	0.277
141390 to 159510	0.278 (average value)
242220 to 246420	0.273 (average value)

the difference between the maximum and minimum values is 0.02K. To increase the accuracy of the temperature difference data, I subtracted the average value 0.272K from each measured value of ΔT. Given that the actual calibration difference between the thermometers at any given time could be anywhere in the range in Table II, I applied uniform distribution with a full width equal to 0.02K to calculate standard uncertainty for ΔT (see Eq. C2), which yields a value of 0.006 for low temperatures.

However, for both types of fibers, at the temperatures for which the value of κ peaked (around 30K), the ΔT readings began fluctuating with differences of around 0.03K. Given a similar frequency to the fluctuations of the pulse tube, it is likely that these fluctuations are driven by the pulse tube temperature oscillations. When encountered, these were treated as uniform distributions (see equation C2), and they were added to the value of uncertainty using the root sum square method:

$$u_c = \sqrt{\sum u_i^2} \tag{C3}$$

where i is an index such that the sum includes all contributions to the uncertainty. The final values varied depending on how much the temperature reading was fluctuating.

f. Average Temperature

The standard uncertainty for average temperature was usually 0.3K for the 1.6mm TP 2HC fiber, and usually 0.14K for the 1.6mm G 2HM fiber. These values are simply half the average calibration differences from Table I and Table II, because the real temperature value could fall anywhere within that difference range, assuming that at least one of the thermometers is close to the true value. At high temperatures, this value was combined with the extra uncertainty from temperature fluctuations using the root sum square method (see equation C3), but this was usually negligible in comparison, except in the high temperature range (above 70K).

2. Uncertainty Propagation

I propagated the uncertainty through calculations using the following:

$$u_c^2 = \sum \left(\frac{\delta f}{\delta y_i} \right)^2 u^2(y_i) \tag{C4}$$

where f is the function, y_i are the individual variables, and u are the uncertainties. This was only necessary when calculating κ with equation 3. The expression used to calculate uncertainty was:

$$U = \sqrt{\left(\frac{\partial \kappa}{\partial I}\right)^2 (u_I)^2 + \left(\frac{\partial \kappa}{\partial V}\right)^2 (u_V)^2 + \left(\frac{\partial \kappa}{\partial L}\right)^2 (u_L)^2 + \left(\frac{\partial \kappa}{\partial \Delta T}\right)^2 (u_{\Delta T})^2 + \left(\frac{\partial \kappa}{\partial A}\right)^2 (u_A)^2} \tag{C5}$$

where the values u represent different values of uncertainty. Because of the nature of the equation, uncertainty varied for the different measured values, from about 90W/m/K at T_{avg} = 8K to nearly 250 W/m/K at T_{avg} = 30K for the TP 2HC fiber, and from about 90W/m/K at T_{avg} = 7.3K to above 180 W/m/K at T_{avg} = 31K for the G 2HM fiber.